

# Sufficient conditions for stabilizability of switched linear systems with sub-optimal convergence rate

A. Jajarmi, N. Pariz, and A. Karimpour

**Abstract**— In this paper, by use of the properties of matrix measure, a set of linear and quadratic conditions is obtained which is sufficient for stabilizability of switched linear systems. These conditions are easily applicable because of their special form. So they are suitable to apply specially for higher order systems with several subsystems. Moreover, different system performances can also be achieved by using an appropriate objective function subject to the proposed conditions. In this way, an optimization problem is also introduced to stabilize the system with sub-optimal convergence rate. Finally a numerical example is used to show the effectiveness of the proposed approach.

**Keywords:** Switched linear system; Stability; Matrix measure; Linear and quadratic conditions;

## I. INTRODUCTION

In past decades, applying hybrid systems has raised considerably in different fields such as manufacturing systems [1,2], communication networks and traffic control [3,4,5], chemical processing [6], and aircraft control [7]. However, Study of hybrid systems is still challenging and in the elementary stage since hybrid models are generally complex [8]. A hybrid system consists of both continuous dynamics and discrete elements to form a dynamic system. A standard model for such systems is presented in [8]. As a special class of hybrid systems, switched systems can be mentioned which contain subsystems and a switching rule which manages the switching between them. Recent years, an increasing interest has been met in analysis and control of switched systems with their applications [9-16].

Constructing a switching law which makes the switched system asymptotically stable is a basic problem in stability and design of switched systems [11]. Some necessary conditions has been provided in [15] for stability of switched linear systems. Ref. [16] dealt with an effort on the synthesis of a switching strategy to stabilize switched linear systems with unstable subsystems. In addition, an state dependent switching strategy was proposed and it has been shown that

the existence of Hurwitz linear convex combination of subsystem matrices is sufficient for the existence of such stabilizing switching strategy. The same assumption of the existence of stable convex combination of subsystems has been met in [17] to present an extension of dynamic output feedback with a robust detectability condition. In this work, a set of conditions has been provided which can guarantee the stability along with quadratic Lyapunov functions via dynamic output feedback. Stability of switched systems with fully controllable subsystems is discussed in [18]. But it doesn't consider the design procedure. Practical stabilization problem for switched linear systems is studied in [19]. In addition, under the same assumption of the existence of stable convex combination, a periodically switching strategy has been constructed which makes the overall system asymptotically stable.

This paper is focused on setting up a set of linear and quadratic conditions for stability of switched linear systems with desirable convergence rate. Besides, some system performances can be obtained by adding appropriate objective functions to the proposed constraints.

The paper is organized as follows. Section 2 contains the preliminaries, section 3 is the main ideas, section 4 gives proposed formulation and a numerical example is presented in section 5.

## II. PRELIMINARIES

Consider the following autonomous switched linear time invariant system:

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(t_0) = x_0 \quad (1)$$

where,  $x(t) \in R^n$  is the state initialized at  $x_0$ ,  $\sigma(t): R^+ \rightarrow \{1, 2, \dots, m\}$  is the piecewise constant switching signal to be designed and  $A_i \in R^{n \times n}$  is real constant matrix which represents the  $i^{th}$  subsystem for  $i \in \{1, 2, \dots, m\}$ . Note that  $\sigma(t) = i$  means the  $i^{th}$  subsystem is active.

Now we introduce the notion of switching sequence which is needed to describe switching signals. A switching sequence over time interval  $[t_0, t_f)$  is defined as

$$\{(t_0, i_0), (t_1, i_1), \dots, (t_M, i_M)\} \quad (2)$$

where,  $t_k, k = 0, 1, \dots, M$  are switching time instants such that  $t_0 < t_1 < \dots < t_M < t_f$  and  $i_k$  is the index of subsystem which is activated at  $t_k$  for  $k = 0, 1, \dots, M$ . Also an associated periodic switching signal with period of  $T = t_f - t_0$  can be defined as follows

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$$\sigma(t) = i_k \quad (3)$$

at time instants  $aT + t_k$  for  $a = 0, 1, \dots$  and  $k = 0, 1, \dots, M$ .

**Definition 2.1** (*stabilizability via periodic switching*):

System (1) is said to be asymptotically stabilizable via periodic switching if there is a periodic switching signal  $\sigma(t)$  such that the system is asymptotically stable.

**Definition 2.2:**

System (1) is said to be (exponentially) convergent if there exist two positive real numbers  $\alpha$  and  $\beta$  such that

$$\|x(t)\| \leq \beta e^{-\alpha(t-t_0)} \quad \forall t \geq t_0 \quad (4)$$

where,  $\alpha$  is known as convergence rate of switched system (1).

### III. MAIN IDEAS

First, the average approach is introduced which is useful to manage the convergence rate of switched system (1):

**Definition 3.1:**

For switched system (1), *average system* can be defined as following linear time invariant system

$$\dot{x}(t) = Ax(t), x(t_0) = x_0 \quad (5)$$

where, matrix  $A$  is linear convex combination of  $A_i: i = 1, 2, \dots, m$ . In addition, if average system is stable, then its convergence rate can be defined as

$$\psi = -\max_{i=1, \dots, n} \{Re \lambda_i(A)\} \quad (6)$$

where,  $\lambda_i(A): i = 1, 2, \dots, n$  are eigen values of  $A$ .

Now a sufficient condition is introduced for stabilizability of switched system (1) via periodic switching:

**Proposition 3.1** [20]:

Suppose that there are real numbers  $w_i \geq 0, i = 1, 2, \dots, m, \sum_{i=1}^m w_i = 1$  such that for system (1)  $A \stackrel{\text{def}}{=} \sum_{i=1}^m w_i A_i$  is Hurwitz. Then the switched system is asymptotically stable via following periodic switching signal:

$$\forall t \geq t_0$$

$$\sigma(t) = \begin{cases} 1 & \text{if } \text{mod}(t, T) \in [0, w_1 T) \\ 2 & \text{if } \text{mod}(t, T) \in [w_1 T, (w_1 + w_2) T) \\ \vdots & \\ m & \text{if } \text{mod}(t, T) \in \left[ \left( \sum_{i=1}^{m-1} w_i \right) T, T \right) \end{cases} \quad (7)$$

where,  $T$  is period of switching which should be chosen small enough and  $\text{mod}(t, T)$  denotes the remainder of  $t$  divided by  $T$ . Moreover, the convergence rate of the switched system can arbitrarily approach that of the average system via periodic switching (7) if period of switching is sufficiently small.

**Proof:**

The solution of the switched system via presented periodic switching after first period can be computed to be

$$x(T) = e^{T w_m A_m} \dots e^{T w_2 A_2} e^{T w_1 A_1} x_0 \quad (8)$$

and after  $a^{\text{th}}$  period is

$$x(aT) = e^{T w_m A_m} \dots e^{T w_2 A_2} e^{T w_1 A_1} x((a-1)T) \quad (9)$$

So the state transition matrix of the system after one period is

$$\varphi = e^{T w_m A_m} \dots e^{T w_2 A_2} e^{T w_1 A_1} \quad (10)$$

On the other hand, since  $A = \sum_{i=1}^m w_i A_i$  is Hurwitz, there exists a sufficiently small  $\varepsilon > 0$  such that  $\varphi$  is Schur stable for any  $T < \varepsilon$  [20]. Hence, by choosing small enough  $T$ , the switched system is asymptotically stable. Latter case is also proved in [20]. ■

This proposition implies that the existence of stable linear convex combination of subsystems is sufficient for switched system (1) to be periodically stabilizable with convergence rate near that of its average system. Note that the same assumption of stable convex combination can be met in [16, 17, and 19]. In this paper, we are interested in finding corresponding set of linear and quadratic conditions which can be verified the existence of stable linear convex combination effectively. Hence, the concept of matrix measure is introduced.

**Definition 3.2:**

Let  $\|\cdot\|_i$  be an induced matrix norm on  $R^{n \times n}$ . Then the corresponding matrix measure is defined by:

$$\begin{cases} \mu(\cdot): R^{n \times n} \rightarrow R \\ \mu(A) = \lim_{h \rightarrow 0^+} \frac{\|I + hA\|_i - 1}{h} \end{cases} \quad (11)$$

In addition, the matrix measures of  $A = (a_{ij})$  corresponding to the two common norms  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  can be formulated as:

$$\mu_1(A) = \sup_k \left( a_{kk} + \sum_{\substack{j=1 \\ j \neq k}}^n |a_{jk}| \right) \quad (12)$$

and

$$\mu_\infty(A) = \sup_j \left( a_{jj} + \sum_{\substack{k=1 \\ k \neq j}}^n |a_{jk}| \right) \quad (13)$$

**Lemma 3.1:**

Let  $\lambda$  be an eigenvalue of matrix  $A$ . Then  $Re \lambda \leq \mu(A)$ .

**Proof.** See [21]. ■

**Corollary 3.1:**

Matrix  $A$  is stable if  $\mu(A) < 0$  for some matrix measure  $\mu$ .

IV. PROPOSED FORMULATION

For switched system (1), let

$$A_i = \begin{bmatrix} a_{11}^i & \cdots & a_{1n}^i \\ \vdots & \ddots & \vdots \\ a_{n1}^i & \cdots & a_{nn}^i \end{bmatrix}, i = 1, 2, \dots, m \quad (14)$$

Then convex combination of subsystem matrices is given by

$$A = \sum_{i=1}^m w_i A_i = \begin{bmatrix} \sum_{i=1}^m a_{11}^i w_i & \cdots & \sum_{i=1}^m a_{1n}^i w_i \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^m a_{n1}^i w_i & \cdots & \sum_{i=1}^m a_{nn}^i w_i \end{bmatrix} \quad (15)$$

where,  $w_i \geq 0, i = 1, 2, \dots, m, \sum_{i=1}^m w_i = 1$ .

Now consider  $\mu_1(A)$  which is obtained as

$$\mu_1(A) = \sup_k \left( \sum_{i=1}^m a_{kk}^i w_i + \sum_{\substack{j=1 \\ j \neq k}}^n \left| \sum_{i=1}^m a_{jk}^i w_i \right| \right) \quad (16)$$

Although the objective function of  $\mu_1(A)$  is nonlinear, it can be transform to a linear function subject to linear and quadratic constraints as follows:

Let  $R_{jk} = \max\{\sum_{i=1}^m a_{jk}^i w_i, 0\}$  and  $S_{jk} = \max\{-\sum_{i=1}^m a_{jk}^i w_i, 0\}$  for  $j, k = 1, 2, \dots, n, j \neq k$ . So either  $R_{jk}$  or  $S_{jk}$  is zero. Then  $\mu_1(A)$  can be rewritten as follows:

$$\begin{cases} \mu_1(A) = \sup_k \left( \sum_{i=1}^m a_{kk}^i w_i + \sum_{\substack{j=1 \\ j \neq k}}^n (R_{jk} + S_{jk}) \right) \\ s. t. \\ R_{jk} S_{jk} = 0 \quad \text{for } j, k = 1, 2, \dots, n, j \neq k \\ R_{jk} - S_{jk} = \sum_{i=1}^m a_{jk}^i w_i \quad \text{for } j, k = 1, 2, \dots, n, j \neq k \\ R_{jk} \geq 0, S_{jk} \geq 0 \quad \text{for } j, k = 1, 2, \dots, n, j \neq k \end{cases} \quad (17)$$

By similar procedure for  $\mu_\infty(A)$ , it is obtained that:

$$\begin{cases} \mu_\infty(A) = \sup_j \left( \sum_{i=1}^m a_{jj}^i w_i + \sum_{\substack{k=1 \\ k \neq j}}^n (R_{jk} + S_{jk}) \right) \\ s. t. \\ R_{jk} S_{jk} = 0 \quad \text{for } j, k = 1, 2, \dots, n, j \neq k \\ R_{jk} - S_{jk} = \sum_{i=1}^m a_{jk}^i w_i \quad \text{for } j, k = 1, 2, \dots, n, j \neq k \\ R_{jk} \geq 0, S_{jk} \geq 0 \quad \text{for } j, k = 1, 2, \dots, n, j \neq k \end{cases} \quad (18)$$

Now the following theorems can be stated:

**Theorem 4.1:**

Suppose that for system (1) the following set of linear and quadratic conditions are satisfied:

$$\begin{cases} \sum_{i=1}^m a_{kk}^i w_i + \sum_{\substack{j=1 \\ j \neq k}}^n (R_{jk} + S_{jk}) < -\alpha, \quad k = 1, 2, \dots, n \\ R_{jk} S_{jk} = 0, \quad k = 1, 2, \dots, n, j \neq k \\ R_{jk} - S_{jk} - \sum_{i=1}^m a_{jk}^i w_i = 0, \quad k = 1, 2, \dots, n, j \neq k \\ R_{jk} \geq 0, S_{jk} \geq 0, \quad j, k = 1, 2, \dots, n, j \neq k \\ \sum_{i=1}^m w_i = 1 \\ w_i \geq 0, \quad i = 1, 2, \dots, m \end{cases} \quad (19)$$

where,  $\alpha > 0$  is desirable convergence rate. Then the system is asymptotically stabilizable with convergence rate better than  $\alpha$ .

**Proof.**

Since  $A = \sum_{i=1}^m w_i A_i$ , satisfaction of last two conditions implies that  $A$  is linear convex combination of subsystem matrices. On the other hand, based on formulation of  $\mu_1(A)$ , satisfaction of first four conditions imply that  $\mu_1(A) < -\alpha$  and hence, by corollary 3.1,  $A$  is stable. Moreover, based on lemma 3.1, since  $Re \lambda(A) < \mu_1(A) < -\alpha$ , for average system we have  $\psi > \alpha$ . Hence, by proposition 3.1, switched system (1) is asymptotically stabilizable with convergence rate better than  $\alpha$ . ■

**Theorem 4.2:**

Suppose that for system (1) the following set of linear and quadratic conditions are satisfied:

$$\left\{ \begin{array}{l} \sum_{i=1}^m a_{jj}^i w_i + \sum_{\substack{k=1 \\ k \neq j}}^n (R_{jk} + S_{jk}) < -\alpha, \quad j = 1, 2, \dots, n \\ R_{jk} S_{jk} = 0, \quad k = 1, 2, \dots, n, j \neq k \\ R_{jk} - S_{jk} - \sum_{i=1}^m a_{jk}^i w_i = 0, \quad k = 1, 2, \dots, n, j \neq k \\ R_{jk} \geq 0, S_{jk} \geq 0, \quad j, k = 1, 2, \dots, n, j \neq k \\ \sum_{i=1}^m w_i = 1 \\ w_i \geq 0, \quad i = 1, 2, \dots, m \end{array} \right. \quad (20)$$

where,  $\alpha > 0$  is desirable convergence rate. Then the system is asymptotically stabilizable with convergence rate better than  $\alpha$ .

**Proof.**

Proof is the same as previous theorem except that  $\mu_\infty(A)$  is used instead of  $\mu_1(A)$ . ■

These theorems present two sets of linear and quadratic conditions for stabilizability of switched linear autonomous systems with desirable convergence rate. It should be note that these criteria are more valuable in the presence of higher order switched systems with several subsystems since they can be easily verified. The other advantage is that different system performances can also be achieved by introducing an appropriate objective function subject to the proposed constraints. For example the following optimization problem can be used to improve the convergence rate of system (1) via periodic switching (7):

$$\left\{ \begin{array}{l} \max \alpha \\ s. t. \\ \text{constraint set (19) or (20)} \end{array} \right. \quad (21)$$

where,  $\alpha$  is a lower bound of convergence rate. Moreover, obtained convergence rate is sub-optimal, since switching law is restricted to be periodic.

V. NUMERICAL EXAMPLE

Here a numerical example is presented to illustrate the effectiveness of the proposed approach in stability analysis of switched linear systems:

Consider a 3<sup>rd</sup> order autonomous switched linear system with three unstable subsystems as:

$$A_1 = \begin{bmatrix} -3 & 0 & -0.25 \\ 0.1 & -4 & -0.5 \\ -1 & 1 & 1 \end{bmatrix}; \quad (22)$$

$$A_2 = \begin{bmatrix} 2 & 1 & 0.2 \\ 0 & 0 & 1 \\ 1 & -1 & -5 \end{bmatrix}; \quad (23)$$

$$A_3 = \begin{bmatrix} -2 & 1 & 0.5 \\ 0 & 1 & 1 \\ 0.5 & 1 & -3 \end{bmatrix} \quad (24)$$

Using theorem 4.1, the following set of conditions is obtained:

$$\left\{ \begin{array}{l} -3w_1 + 2w_2 - 2w_3 + (R_{21} + S_{21}) + (R_{31} + S_{31}) + \alpha < 0 \\ -4w_1 + w_3 + (R_{12} + S_{12}) + (R_{32} + S_{32}) + \alpha < 0 \\ w_1 - 5w_2 - 3w_3 + (R_{13} + S_{13}) + (R_{23} + S_{23}) + \alpha < 0 \\ -(0.1w_1) + R_{21} - S_{21} = 0 \\ -(-w_1 + w_2 + 0.5w_3) + R_{31} - S_{31} = 0 \\ -(w_2 + w_3) + R_{12} - S_{12} = 0 \\ -(w_1 - w_2 + 2w_3) + R_{32} - S_{32} = 0 \\ -(-0.25w_1 + 0.2w_2 + 0.5w_3) + R_{13} - S_{13} = 0 \\ -(-0.5w_1 + w_2 + w_3) + R_{23} - S_{23} = 0 \\ R_{jk} S_{jk} = 0, \quad j, k = 1, 2, 3, \quad j \neq k \\ R_{jk} \geq 0, S_{jk} \geq 0, \quad j, k = 1, 2, 3, \quad j \neq k \\ w_1 + w_2 + w_3 = 1 \\ w_i \geq 0, \quad i = 1, 2, 3 \\ \alpha > 0 \end{array} \right. \quad (25)$$

Now, use proposed strategy to stabilize the system with sub-optimal convergence rate. So the following optimization problem is obtained:

$$\left\{ \begin{array}{l} \max \alpha \\ s. t. \\ \text{constraint set (25)} \end{array} \right. \quad (26)$$

The optimal solution is

$$\alpha = 1.1774, w_1 = 0.5549, w_2 = 0.2828, w_3 = 0.1624.$$

Note that convergence rate of average system is  $\psi = 1.3152$ . So the system is asymptotically stabilizable via periodic switching (7) with convergence rate better than  $\alpha = 1.1774$ .

Figure 1 depicts the state trajectory of switched system initialized at  $x_0 = [0.3, 0.2, 0.1]$  via periodic switching:

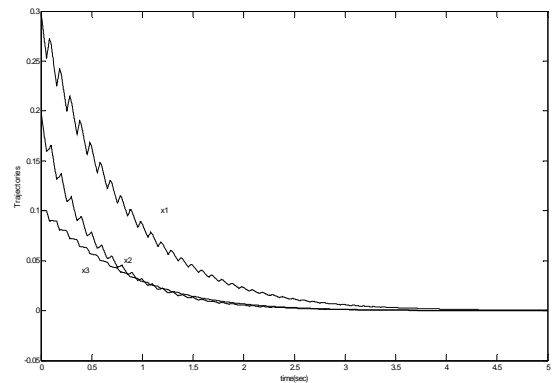


Figure 1. State trajectories under periodic switching with convergence rate better than  $\alpha = 1.1774$

Figure 2 shows part of the periodic switching signal:

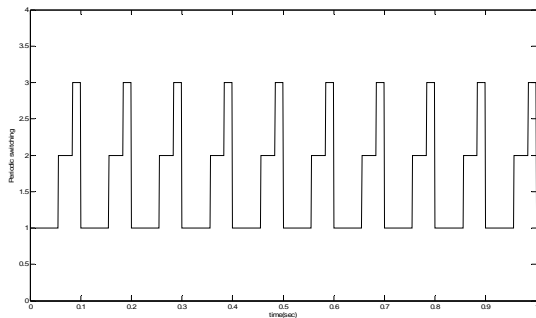


Figure 2. Part of periodic switching signal

## VI. CONCLUSION

In this paper, properties of matrix measure have been used to obtain a set of linear and quadratic conditions for stability of switched linear systems with desirable convergence rate. These conditions need no tedious arithmetic to apply and thus easy to put into practice. In addition, they are more valuable in the presence of higher order systems with several subsystems since they can be easily verified. Besides, some system performances can be added by introducing appropriate objective functions subject to these conditions. As an example, an optimization problem has also been introduced to stabilize the system with sub-optimal convergence rate.

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